The Dirac Operator over Abelian Finite Groups

Jayme Vaz, Jr.[1]

Department of Physics, Syracuse University
Syracuse NY 13244-1130 USA

Abstract

In this paper we show how to construct a Dirac operator on a lattice in complete analogy with the continuum. In fact we consider a more general problem, that is, the Dirac operator over an abelian finite group (for which a lattice is a particular example). Our results appear to be in direct connexion with the so called fermion doubling problem. In order to find this Dirac operator we need to introduce an algebraic structure (that generalizes the Clifford algebras) where we have quantities that work as square-root of the translation operator. Quantities like these square-roots have been used recently in order to provide an approach to fermions on the lattice that is free from doubling and has chiral invariance in the massless limit, and our studies seem to give a mathematical basis to it.

1 Introduction

As well-known, the usual study of fermions in lattice field theories (LFT) is defective. Of course one of the major problems is the so called fermion doubling problem[1]. We feel, however, that there is another big disappointment concerning LFT. In fact, while in the continuum the Dirac operator can be written in the form $\partial = \sum_\mu \gamma^\mu \partial_\mu$ as a square root of the Laplace-Beltrami operator $\Box$, that is $\partial^2 = \Box$, the same appears to not happen on the lattice. Indeed, to the best of our knowledge, we don't have an expression for the Dirac operator on the lattice that is of the form $\sum_\mu \Gamma^\mu D_\mu$, where Γ^μ are operators (like the gamma matrices in the continuum) and D_μ are derivatives. The origin of this problem is that while in the continuum the laplacian is given by $\sum_\mu \partial^\mu \partial_\mu$, on a lattice we need both the forward and backward derivatives, that is, the laplacian is given by $\sum_\mu D_\mu^+ D_\mu^- = \sum_\mu D_\mu^- D_\mu^+$. The problem,

[1]On leave of absense from Department of Applied Mathematics, State University at Campinas, Campinas, SP, 13081-970, Brazil. E-mail: vaz@ime.unicamp.br

therefore, is if it exists an operator on a lattice that is of the form $\sum_\mu \Gamma^\mu D^-_\mu$ or $\sum_\mu \Gamma^\mu D^+_\mu$ such that its square gives the laplacian $\sum_\mu D^+_\mu D^-_\mu = \sum_\mu D^-_\mu D^+_\mu$.

In a previous work[2] we have tried to solve this important problem. Our approach was along the direction of Dirac-Kähler spinor fields (DKSF)[3] in the sense that the gamma matrices can be thought as operators acting on differential forms. In terms of the DKSF we also have the remarkable relationship $\partial = \mathrm{d} - \delta$, where ∂ is the Dirac operator and d and δ are the usual differential and (Hodge) co-differential operators, respectively[4]. Note that this is not a definition since ∂ is defined using a Clifford algebra while d and δ are defined using the exterior (or Grassmann) algebra. Our approach, however, had some limitations, particularly in relation to the geometry of the lattice (we succeed to find an answer for some particular cases) and we have been looking since then for a more satisfactory approach. Our purpose in this paper is to present such an approach, that is, to present a formalism such that we have an operator ∂ over a lattice such that its square ∂^2 gives the lattice transcription of the Laplace-Beltrami operator, with these operators acting on lattice analogues of DKSF.

Of course to look for such a "true" lattice analogue of the Dirac operator is a problem which is important by itself. However, this could be the key to the solution of the fermion doubling problem, as shown by Feng, Li and Song[5]. These authors arrived from a different approach to an operator like the one once introduced in [2] (whose generalization we shall discuss in this paper) and it seems that the approach in terms of this operator is free from the doubling problem. Since our approach and the one of [5] are very different, studies remain to be done to shown the relationship (if any) between those operators, as well as if it really provide a solution to that old problem, but anyway it is amazing that such a relationship seems to exist and that it could provide a solution to the doubling problem.

In this paper we shall introduce the version of the Dirac operator over an abelian finite group G. A d-dimensional lattice is a particular case for which G is of the form $G = \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_d}$. Our approach in terms of an arbitrary abelian finite group removes therefore several limitations related to the geometry of a particular lattice, which is welcome. We organized this paper as follows. In section 2 we briefly introduce some basic mathematical tools from noncommutative geometry that will be needed. Most of the materials in section 2 are well-known and not restricted to abelian finite groups, but since this is the case we are interested we shall restrict our attention to it. Some references are[6, 7, 8, 9, 10]. In section 3 we discuss the basics of the

exterior algebra, obtaining some preliminary results that will be needed in the sequel. In section 4 we introduce a generalized Clifford algebra, which is the algebraic structure we need in order to construct the Dirac operator. This Clifford-like algebra is a generalization of the Clifford algebras of the continuum. It is no surprise that we need such a generalized Clifford algebra in order to construct the Dirac operator. Indeed, it is reasonable to suppose that if the Dirac operator over an abelian finite group is different from its continuum version, then the algebraic structure needed to construct it must also be different from its continuum version. Like this version of Dirac operator is expected to reduce to the usual one in the continuum limit, this generalized Clifford algebra is expect to reduce to the usual one in the same limit - as this will be the case indeed. Then, using this generalized Clifford algebra, we introduce the Dirac operator over an abelian finite group. In section 5 we show how these results apply to LFT and to the fermion doubling problem.

2 Calculus over Finite Groups

We are interested in a calculus over an abelian finite group, and noncommutative geometry provides the tools we need. We shall consider a finite group G with elements $x, y, z, \ldots$, which we suppose to be abelian. In fact, the hypothesis of an abelian group will be needed only in the following sections – the results of this section also apply to non-abelian groups[11] – but since we don't see any advantage in leaving this hypothesis for the next section, we shall suppose it from the beginning.

Let $\mathcal{A}$ be the algebra of functions in G with values in $\mathbb{C}$ or $\mathbb{R}$. An arbitrary function $f \in \mathcal{A}$ can be written in the form

$$f = \sum_{x \in G} f(x) \mathbf{e}^x, \tag{1}$$

where $f(x) \in \mathbb{C}$ and $\mathbf{e}^x$ is such that

$$\mathbf{e}^x(y) = \delta^x_y. \tag{2}$$

The unit $1_{\mathcal{A}}$ can be written as $1_{\mathcal{A}} = \sum_x \mathbf{e}^x$.

The 1-forms are elements of Ω^1, which is generated as an $\mathcal{A}$-bimodule. The differential operator $\mathrm{d} : \Omega^0 = \mathcal{A} \to \Omega^1$ is defined as

$$\mathrm{d}f = 1 \otimes f - f \otimes 1. \tag{3}$$

If we define $\mathbf{e}^{x,y} = \mathbf{e}^x \otimes \mathbf{e}^y$ for $x \neq y$ and $\mathbf{e}^{x,x} = 0$ we can write

$$\mathrm{d}\mathbf{e}^x = \sum_y (\mathbf{e}^{y,x} - \mathbf{e}^{x,y}). \tag{4}$$

Note that $f\mathrm{d}g \neq (\mathrm{d}g)f$, that is, the calculus is non-commutative, even if $\mathcal{A}$ is a commutative algebra. This is the so called universal first-order differential calculus (UFODC)[7]. By universality we mean that any other first-order differential calculus can be obtained from the UFODC by an appropriated quotient. This corresponds to cases where some of those $\mathbf{e}^{x,y}$ may vanish for $x \neq y$. For the UFODC if there is an involution $*$ in $\mathcal{A}$ it can be extended to Ω^1 as $(f \otimes g)^* = g^* \otimes f^*$, but this don't need to be the case for an arbitrary first-order differential calculus.

The universal differential calculus (UDC) is (d, Ω), where $\Omega = \oplus_{k=0}^{\infty} \Omega^k$, with Ω^k being the space of k-forms and $\mathrm{d} : \Omega^k \to \Omega^{k+1}$. The space Ω^k is given by the tensor product (over $\mathbb{C}$) of $k+1$ copies of $\mathcal{A} = \Omega^0$, which we denote by $\Omega^k = \mathcal{A}^{\otimes^{k+1}}$. We denote by π the projection $\pi : (\Omega^1)^{\otimes^k} \to \Omega^k$. The operator d can be written as

$$\mathrm{d}\mathbf{e}^{x_1,\cdots,x_k} = \sum_y (\mathbf{e}^{y,x_1,\cdots,x_k} - \mathbf{e}^{x_1,y,x_2,\cdots,x_k} + \cdots (-1)^k \mathbf{e}^{x_1,\cdots,x_k,y}), \tag{5}$$

where $\mathbf{e}^{x_1,\cdots,x_k}$ is zero if any two adjacent indeces are equal and $\mathbf{e}^{x_1} \otimes \cdots \otimes \mathbf{e}^{x_k}$ otherwise. Any differential calculus is obtained from the UDC by an appropriated quotient.

Let us define the translation $\mathcal{R}_x$ as

$$(\mathcal{R}_x f)(y) = f(y + x). \tag{6}$$

We can extend this definition to Ω^1 (and so on) according to $\mathcal{R}_x(f \otimes g) = (\mathcal{R}_x f) \otimes (\mathcal{R}_x g)$. In the case of a nonabelian group we need to distinguish in this case the right and left translations. There is a particularly importante set of 1-forms, which we denote by θ^x, defined by

$$\theta^x = \sum_y \mathbf{e}^{y,y+x}. \tag{7}$$

The importance of these 1-forms is because they are invariant, that is, $\mathcal{R}_x \theta^y = \theta^y$. For nonabelian groups we can define sets of 1-forms that are left and right invariant (in terms of left and right translations) and in general those sets are different[12].

The noncommutative of the present calculus can be expressed through

$$\theta^x f = (\mathcal{R}_{(x)} f)\theta^x, \tag{8}$$

where we used the brackets to explicitly indicate that there is no sum implied in this formula. The differential of a function $f \in \mathcal{A}$ can be expressed now as

$$\mathrm{d}f = \theta f - f\theta = \sum_x (D_x^+ f)\theta^x = \sum_x \theta^x (D_x^- f), \tag{9}$$

where we defined $\theta = \sum_x \theta^x$ and

$$D_x^+ f = \mathcal{R}_x f - f, \qquad D_x^- f = f - \mathcal{R}_x^{-1} f. \tag{10}$$

Now, let us define the dual space to Ω^1, that is, define the vector fields. In the differential geometry of manifolds, vector fields ∂_x (x here is a point of the manifold) can be defined by means of $\mathrm{d}f(\partial_x) = \partial_x f$. This is the definition of a contraction, and we can write it in a more convenient way as $\langle \partial_x, \mathrm{d}f\rangle = \langle \mathrm{d}f, \partial_x\rangle = \partial_x f$, where we used left and right contractions, respectively. In the case of the geometry of manifolds there is no need to distinguish between left and right contractions of 1-forms by vector fields since both give the same result, but this is not the case in noncommutative geometry. It is not difficult to see this, and (to the best of our knowledge) the authors that make explicit use of contractions have chosen one over another[8, 9]. In our oppinion this is not the correct approach since we believe that *both* contractions (left and right) are needed.

We shall, therefore, define and consider both left and right contractions. However, before doing it, a word about the notation is needed. We shall not use a notation like ∂_x for vector fields. We prefer to use instead the notation D_{θ^x}. This would be equivalent to write $D_{\mathrm{d}x}$ for ∂_x in the geometry of manifold, and is remind us of the grassmanian character of the differential $\mathrm{d}x$ (in fact we usually denote by ∂_θ or D_θ the dual quantity of a grassmanian variable θ). In terms of the geometry of manifolds we believe that we don't see the advantages of this notation since almost everyone is used with that other notation, but in the context of noncommutative geometry and of the specific problem we are addressing in this paper we can see only advantages in using this new notation over the old one.

Now we define the left and right contractions of the 1-form $\mathrm{d}f$ by the vector field D_{θ^x} as

$$\langle D_{\theta^x}, \mathrm{d}f\rangle = D_x^- f, \qquad \langle \mathrm{d}f, D_{\theta^x}\rangle = D_x^+ f, \tag{11}$$

respectively. There is a very beautiful characterization of this operations and of the vector fields D_{θ^x} in terms of Hopf algebras, but we shall not discuss this here - see for example [9]. We have the following properties:

$$f\langle D_{\theta^x}, \psi\rangle = \langle f D_{\theta^x}, \psi\rangle = \langle D_{\theta^x}(\mathcal{R}_x f), \psi\rangle = \langle D_{\theta^x}, (\mathcal{R}_x f)\psi\rangle, \tag{12}$$

with an analogous expression for the properties of the right contraction. For more details see [9, 13].

3 Wedge Product and Exterior Algebra

Wonorowicz[12] has shown that for a bicovariant calculus there exists a unique bimodule isomorphism $\Lambda : \Omega^1 \otimes \Omega^1 \to \Omega^1 \otimes \Omega^1$, which satisfies the Yang-Baxter equation – being a bimodule isomorphism it also satisfies

$$\Lambda(f\Phi g) = f\Lambda(\Phi)g, \qquad f, g \in \mathcal{A}, \quad \Phi \in \Omega^1 \otimes \Omega^1. \tag{13}$$

This result can be applied to our present case, and indeed for an abelian finite group we have the simple result

$$\Lambda(\theta^x \otimes \theta^y) = \theta^y \otimes \theta^x. \tag{14}$$

For a nonabelian group, see [11].

The wedge product of 1-forms ψ and ϕ is defined as

$$\psi \wedge \phi = (\pi \circ \mathsf{A})(\psi \otimes \phi), \tag{15}$$

with π being the projection $\Omega^1 \otimes \Omega^1 \to \Omega^2$ and

$$\mathsf{A} = \mathrm{id} \otimes \mathrm{id} - \Lambda. \tag{16}$$

In spite the expression for Λ - eq.(14) - that resembles the ordinary permutation operation, we have in general

$$\psi \wedge \phi \neq -\phi \wedge \psi. \tag{17}$$

This is due to the noncommutativity of functions and 1-forms as in eq.(8). However, we still have one simple expression in our case of an abelian finite group for a particular product, namely

$$\theta^x \wedge \psi = -(\mathcal{R}_x\psi) \wedge \theta^x, \qquad \psi \in \Omega^1 = \bigwedge{}^1. \tag{18}$$

It is not difficult to generalize the wedge product for arbitrary k-forms, and the formulas can be find in [9, 13], so that we shall not reproduce them here. The space of k-forms will be denoted by $\bigwedge^k$. The generalization of eq.(19) is

$$\theta^x \wedge \psi = (\#\mathcal{R}_x \psi) \wedge \theta^x, \tag{19}$$

where # denotes the involution defined by

$$\#\psi_k = (-1)^k \psi_k, \qquad \psi_k \in \bigwedge{}^k. \tag{20}$$

Eq.(19) can be written in a form that will be important for what follows. Indeed we shall look for operators acting over multiforms, and eq.(19) can be interpreted from an operator point of view as follows. Let us define the operators $\mathbf{E}(\theta^x)$ and $\mathbf{E}^\dagger(\theta^x)$ as

$$\mathbf{E}(\theta^x)(\psi) = \theta^x \wedge \psi, \qquad \mathbf{E}^\dagger(\theta^x)(\psi) = \psi \wedge \theta^x, \tag{21}$$

that is, they are left and right wedge multiplications. Eq.(19) then implies that

$$\mathbf{E}(\theta^x) = \mathbf{E}^\dagger(\theta^x)\mathcal{R}_x\#, \qquad \mathbf{E}^\dagger(\theta^x) = \mathbf{E}(\theta^x)\mathcal{R}_x^{-1}\#. \tag{22}$$

Other properties that can be easily verified are

$$\mathbf{E}(\theta^x)\mathbf{E}^\dagger(\theta^y) = \mathbf{E}^\dagger(\theta^y)\mathbf{E}(\theta^x), \tag{23}$$

$$\mathbf{E}(\theta^x)\mathbf{E}(\theta^y) + \mathbf{E}(\theta^y)\mathbf{E}(\theta^x) = 0. \tag{24}$$

If we define $\mathbf{E}(f)(\psi) = f\psi$ and $\mathbf{E}^\dagger(f)(\psi) = \psi f$ we can also write

$$\mathbf{E}(f\theta^x) = \mathbf{E}(f)\mathbf{E}(\theta^x), \qquad \mathbf{E}(\theta^x f) = \mathbf{E}(\theta^x)\mathbf{E}(f), \tag{25}$$

$$\mathbf{E}^\dagger(\theta^x f) = \mathbf{E}^\dagger(f)\mathbf{E}^\dagger(\theta^x), \qquad \mathbf{E}^\dagger(f\theta^x) = \mathbf{E}^\dagger(\theta^x)\mathbf{E}^\dagger(f). \tag{26}$$

This equations must be used with eq.(8).

Our next step is to consider the contractions. The left and right contractions of 1-forms by vector fields given by eq.(11) can be easily to $(\Omega^1)^{\otimes k}$. For example, the generalization of the left contraction to $(\Omega^1)^{\otimes k}$ is given by $\langle D_{\theta^x}, \psi_1 \otimes \cdots \otimes \psi_k\rangle = \langle D_{\theta^x}, \psi_1\rangle \psi_2 \otimes \cdots \otimes \psi_k$. The expression for the right contraction is analogous. The extension of these contraction to the exterior algebra can now be defined. Let us denote the left contraction by D_{θ^x} in this context by $\mathbf{I}(D_{\theta^x})$. An element of $\bigwedge^k$ is of the form $\psi = (\pi \circ \mathsf{A})(\psi_1 \otimes \cdots \otimes \psi_k)$,

where $\pi = \pi_k$ now must be the projection $(\Omega^1)^{\otimes^k} \to \Omega^k$ and $\mathsf{A} : (\Omega^1)^{\otimes^k} \to (\Omega^1)^{\otimes^k}$ is the appropriated generalization of that A given by eq.(16) - see [9, 13]. The left contraction $\mathbf{I}(D_{\theta^x}) : \bigwedge^k \to \bigwedge^{k-1}$ can be defined as

$$\mathbf{I}(D_{\theta^x})(\pi_k[\mathbf{A}(\psi_1 \otimes \cdots \otimes \psi_k)]) = \pi_{k-1}[\langle D_{\theta^x}, \mathbf{A}(\psi_1 \otimes \cdots \otimes \psi_k)\rangle]. \tag{27}$$

The expression for the right contraction, which we denote by $\mathbf{I}^\dagger(\theta^x)$, is defined analogously.

From the definition one can prove the following important properties

$$\mathbf{I}(D_{\theta^x})(\psi \wedge \phi) = [\mathbf{I}(D_{\theta^x})(\psi)] \wedge \phi + (\#\mathcal{R}_x^{-1}\psi) \wedge [\mathbf{I}(D_{\theta^x})(\phi)], \tag{28}$$
$$\mathbf{I}^\dagger(D_{\theta^x})(\psi \wedge \phi) = \psi \wedge [\mathbf{I}^\dagger(D_{\theta^x})(\phi)] + [\mathbf{I}^\dagger(D_{\theta^x})(\psi)] \wedge (\#\mathcal{R}_x\phi). \tag{29}$$

The contraction with an arbitrary vector field can be calculated using $\mathbf{I}(fD_{\theta^x}) = \mathbf{E}(f)\mathbf{I}(D_{\theta^x})$, $\mathbf{I}(D_{\theta^x}f) = \mathbf{I}(D_{\theta^x})\mathbf{E}(f)$ and $fD_{\theta^x} = D_{\theta^x}\mathcal{R}_x f$, with analogous formulas for the right contraction. Properties analogous to eqs.(28,29) obviously don't hold for an arbitrary vector field since $f\psi \neq \psi f$. However, these properties are enough for our purposes.

From eqs.(28,29) it follows the following important property:

$$\mathbf{I}^\dagger(D_{\theta^x}) = -\mathbf{I}(D_{\theta^x})\mathcal{R}_x\#, \qquad \mathbf{I}(D_{\theta^x}) = -\mathbf{I}^\dagger(D_{\theta^x})\mathcal{R}_x^{-1}\#. \tag{30}$$

Moreover, left and right contractions by D_{θ^x} commute, that is,

$$\mathbf{I}(D_{\theta^x})\mathbf{I}^\dagger(D_{\theta^y}) = \mathbf{I}^\dagger(D_{\theta^y})\mathbf{I}(D_{\theta^x}). \tag{31}$$

From these last two equations it follows that

$$\mathbf{I}(D_{\theta^x})\mathbf{I}(D_{\theta^y}) + \mathbf{I}(D_{\theta^y})\mathbf{I}(D_{\theta^x}) = 0, \tag{32}$$

with an analogous equation for the right contraction. This equation again only holds for the vector fields $\{D_{\theta^x}\}$ and not for arbitrary ones (since $\{\theta^x\}$ are (left and right) translation invariants).

There are some formulas that follow from eqs.(28,29) that will be of interest for us. In particular we have

$$\mathbf{I}(D_{\theta^x})\mathbf{E}(\theta^y) + \mathbf{E}(\theta^y)\mathbf{I}(D_{\theta^x}) = \delta_x^y, \tag{33}$$
$$\mathbf{I}^\dagger(D_{\theta^x})\mathbf{E}(\theta^y) - \mathbf{E}(\theta^y)\mathbf{I}^\dagger(D_{\theta^x}) = \delta_x^y\mathcal{R}_x\#. \tag{34}$$

4 Generalized Clifford Algebras and the Dirac Operator

Clifford algebras can be defined in several different ways. One of these ways is as a subalgebra of the algebra of endomorphisms of the exterior algebra. Let us considerer the geometry of manifolds; using a notation analogous to the one of last section, if we denote the wedge multiplication by $\mathrm{d}x^\mu$ as $\mathbf{E}(\mathrm{d}x^\mu)$ and the left contraction by the vector field ∂_μ as $\mathbf{I}(\partial_\mu)$, then the quantities $\gamma^\mu = \mathbf{E}(\mathrm{d}x^\mu)+\mathbf{I}(\partial_\mu)$ generate a Clifford algebra, that is, γ^μ satisfies $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}$ (in case of an arbitrary inner product we take $\gamma^\mu = \mathbf{E}^(\mathrm{d}x^\mu) + g^{\mu\nu}\mathbf{I}(\partial_\nu))$. The Dirac operator can now be defined as $\partial = \sum_\mu \gamma^\mu\partial_\mu$. We must note, however, that there is also another natural possibility, that is, to consider the quantities $\check{\gamma}^\mu = \mathbf{E}(\mathrm{d}x^\mu) - \mathbf{I}(\partial_\mu)$. These quantities $\check{\gamma}^\mu$ generates a Clifford algebra for a space with opposite inner product, that is, they satisfies $\check{\gamma}^\mu\check{\gamma}^\nu + \check{\gamma}^\nu\check{\gamma}^\mu = -2\delta^{\mu\nu}$. Both algebras (generated by $\{\gamma^\mu\}$ and $\{\check{\gamma}^\mu\}$) are needed in order to describe all endomorphisms of the exterior algebra, but we can in fact consider only one of these algebras! This is because the right multiplication by γ^μ is equivalent to left multiplication by $\check{\gamma}^\mu$ (apart from the involution $\#$). This means that instead of working with $\{\gamma^\mu\}$ and $\{\check{\gamma}^\mu\}$ acting from the left, we can work only with $\{\gamma^\mu\}$ but now acting both from the left and from the right, and satisfying of course the same commutation relations no matter what side they are. This question is treated in details in[14, 15, 16], so we invite the interested reader to see these references for more details since we just need the above ideas for what follows.

We are looking therefore for an operator that is the transcription (in terms of abelian finite groups) of the operator $\partial = \sum_\mu \gamma^\mu\partial_\mu$, whose square is the laplacian $\partial^2 = \sum_\mu \partial^\mu\partial_\mu = \Box$. Let us denote this operator by the same letter since there is no risk of confusion in this case. So, we want an operator ∂ such that

$$\partial^2 = \sum_x D_x^+ D_x^- = \sum_x D_x^- D_x^+ = \Box, \tag{35}$$

where $\Box$ now denotes the transcription of the laplacian for a finite group. In analogy to the continuum, it is reasonable to suppose that ∂ should have the form $\sum_x \Gamma^x D_x^-$ or $\sum_x \Gamma^x D_x^+$. But here we have two problems: first, how we choose among D_x^+ and D_x^- in these expressions, and secondly, who is Γ^x? If

we want eq.(35) to be satisfied, then $\{\Gamma^x\}$ cannot satisfies a Clifford algebra relation $\Gamma^x\Gamma^y + \Gamma^y\Gamma^x = 2\delta^{xy}$. If $\{\Gamma^x\}$ are supposed to satisfy a Clifford algebra relation then it seems that there is only one way to factor the laplacian, that is, to use two operators like $\bar{\partial}^+ = \sum_x \Gamma^x D_x^+$ and $\bar{\partial}^- = \sum_x \Gamma^x D_x^-$, and then $\bar{\partial}^+\bar{\partial}^- = \bar{\partial}^-\bar{\partial}^+ = \Box$. But this is what we are trying to avoid! The natural guess therefore is to suppose that $\{\Gamma^x\}$ does *not* generate a Clifford algebra but instead another algebra that reduces to a Clifford algebra in the continuum limit.

The expression for γ^μ that generates a Clifford algebra in the continuum can be written as $\gamma^\mu = \mathbf{E}(\mathrm{d}x^\mu) + \mathbf{I}(\partial_\mu) = \mathbf{E}(\mathrm{d}x^\mu) - \mathbf{I}^\dagger(\partial_\mu)\#$, where in the second equality we used the relation between left and right contractions that holds in the continuum case, namely $\mathbf{I}(\partial_\mu) = -\mathbf{I}^\dagger(\partial_\mu)\#$ - see [17]. This last expression, however, does *not* hold in our present case, where we have instead the relation given by eq.(30). This means that if we translate the above expressions to our case as $\gamma_1^x = \mathbf{E}(\theta^x) + \mathbf{I}(D_{\theta^x})$ and $\gamma_2^x = \mathbf{E}(\theta^x) - \mathbf{I}^\dagger(D_{\theta^x})\#$ respectively, then these quantities are no longer equivalent. Of course the same happens if we apply the same reasoning to the quantities $\mathbf{E}(\theta^x)$ and $\mathbf{E}^\dagger(\theta^x)$ related by eq.(22) instead of an equation like the one in the continuum, namely $\mathbf{E}(\mathrm{d}x^\mu) = \mathbf{E}^\dagger(\mathrm{d}x^\mu)\#$. In summary, this means that we have some different possibilities concerning the definition of a quantity Γ^x that we hope to solve our problem, and the only criterion we have seen in order to decide for one of these different possible generalizations it to choose the one that works (if any). Fortunately there is one!

We shall define the quantities $\{\Gamma^x\}$ as

$$\Gamma^x = \mathbf{E}(\theta^x) - \mathbf{I}^\dagger(D_{\theta^x})\#. \tag{36}$$

From eq.(30) we can also write

$$\Gamma^x = \mathbf{E}(\theta^x) + \mathbf{I}(D_{\theta^x})\mathcal{R}_x. \tag{37}$$

Note that if we are working in a space with an inner product such that $g(\theta^x, \theta^y) = g^{xy}$ then we need to use instead $g^{xy}\mathbf{I}(D_{\theta^y})$ in the above expressions. The details about the definition of a metric over a finite group can be found in [11].

The above quantities $\{\Gamma^x\}$ does not satisfies a Clifford algebra, but instead a generalized one of the form

$$\Gamma^x\Gamma^y + \Gamma^y\Gamma^x = 2\delta^{xy}\mathcal{R}_x. \tag{38}$$

The above relation follows after using eqs.(33,34) in the above definitions. We see therefore that the quantity Γ^x is the square-root of the translation operator,

$$(\Gamma^x)^2 = \mathcal{R}_x. \tag{39}$$

On the other hand, let us consider the quantities Γ^{*x} given by

$$\Gamma^{\dagger x} = \mathbf{E}^\dagger(\theta^x)\mathcal{R}_x + \mathbf{I}^\dagger(D_{\theta^x}). \tag{40}$$

We can easily see that $\{\Gamma^{\dagger x}\}$ satisfy the same commutation relations,

$$\Gamma^{\dagger x}\Gamma^{\dagger y} + \Gamma^{\dagger y}\Gamma^{\dagger x} = 2\delta^{xy}\mathcal{R}_x. \tag{41}$$

These quantities are related to by $\{\check{\Gamma}^x\}$ defined by

$$\check{\Gamma}^x = \mathbf{E}(\theta^x) + \mathbf{I}^\dagger(D_{\theta^x}) = \mathbf{E}(\theta^x) - \mathbf{I}(D_{\theta^x})\mathcal{R}_x \tag{42}$$

according to

$$\Gamma^{\dagger x} = \check{\Gamma}^x \#. \tag{43}$$

Moreover we have

$$\Gamma^x\Gamma^{\dagger y} - \Gamma^{\dagger y}\Gamma^x = 0. \tag{44}$$

The situation is exactly analogous to the continuum, and indeed for $\{\check{\Gamma}^x\}$ we have

$$\check{\Gamma}^x\check{\Gamma}^y + \check{\Gamma}^y\check{\Gamma}^x = -2\delta^{xy}\mathcal{R}_x. \tag{45}$$

We see therefore that we have now a picture completely analogous to the continuum.

Now comes the Dirac operator. We define the Dirac operator as

$$\partial = \sum_x \Gamma^x D_x^-. \tag{46}$$

It easily follows from eq.(38) that

$$\partial^2 = \sum_x D_x^+ D_x^- = \Box, \tag{47}$$

that is, the square of the Dirac operator defined as above gives the laplacian over an abelian finite group, and this is indeed what we expect from a "true" Dirac operator.

The analogy with the continuum goes further. As well-known [3] we have the relationship $\partial = \mathrm{d} - \delta$, where ∂ is the Dirac operator and d and δ are the differential and the (Hodge) codifferential operators. The action of the operators d and δ can be obtained from the action of the Dirac operator by considering both the left and right actions of it. The Dirac operator $\partial^\dagger$ acting on the right is obviously

$$\partial^\dagger = \sum_x \Gamma^{\dagger x} D_x^- . \tag{48}$$

The differential and codifferential can now be expressed as

$$\mathrm{d}\psi = \frac{1}{2}(\partial\psi + \partial^\dagger \# \psi) \tag{49}$$

and

$$-\delta\psi = \frac{1}{2}(\partial\psi - \partial^\dagger \# \psi), \tag{50}$$

and then

$$\partial = \mathrm{d} - \delta. \tag{51}$$

It is a straightforward calculation to show that the above equations are equivalent to

$$\mathrm{d}\psi = \mathbf{E}(\theta^x)\psi - \mathbf{E}^\dagger(\theta^x)\#\psi \tag{52}$$

and

$$\delta\psi = \mathbf{I}(D_{\theta^x})\psi + \mathbf{I}^\dagger(D_{\theta^x})\#\psi. \tag{53}$$

5 Lattice Field Theories

LFT are an obvious arena of applications for the results of the last section. The group G for 4-dimensional LFT is $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3} \times \mathbb{Z}_{N_4}$ and topologically we have a discretization of a 4-torus. The noncommutative

calculus for LFT corresponds to a reduction of the UDC for an oriented lattice [8], that is, we have $\mathbf{e}^{x,y} \neq 0$ only for $x = (n_1, n_2, n_3, n_4)$ and $y = \{(n_1+1, n_2, n_3, n_4), (n_1, n_2+1, n_3, n_4), (n_1, n_2, n_3+1, n_4), (n_1, n_2, n_3, n_4+1)\}$. Let us denote $\hat{1} = (1,0,0,0), \hat{2} = (0,1,0,0), \hat{3} = (0,0,1,0), \hat{4} = (0,0,0,1)$. We have four 1-forms $\theta^{\hat{\mu}}$ ($\mu = 1,2,3,4$), given by

$$\theta^{\hat{\mu}} = \sum_{(n_1,n_2,n_3,n_4)} \mathbf{e}^{(n_1,n_2,n_3,n_4),(n_1,n_2,n_3,n_4)+\hat{\mu}}. \tag{54}$$

The four quantities $\Gamma^{\hat{\mu}}$ are given by

$$\Gamma^{\hat{\mu}} = \mathbf{E}(\theta^{\hat{\mu}}) + \mathbf{I}(D_{\theta^{\hat{\mu}}})\mathcal{R}_{\hat{\mu}} \tag{55}$$

and they satisfy

$$\Gamma^{\hat{\mu}}\Gamma^{\hat{\nu}} + \Gamma^{\hat{\nu}}\Gamma^{\hat{\mu}} = 2\delta^{\mu\nu}\mathcal{R}_{\hat{\mu}}. \tag{56}$$

The Dirac operator is

$$\partial = \sum_{\mu} \Gamma^{\hat{\mu}} D^{-}_{\hat{\mu}}. \tag{57}$$

The above results appear to be directly related to the so-called fermion doubling problem. Recently Feng, Li and Song (FLS) [5] provide a formulation of LFT that seems to be free from fermion doubling and with chiral invariance in the massless limit. The idea behind the FLS approach is to define operators corresponding to half-spacing translation. Since there is no meaning to a half-space translation in the real space, FLS defined it in the momentum space. They defined operators $\mathcal{R}_{\mu/2}$ satisfying (i) $\mathcal{R}^2_{\mu/2} = \mathcal{R}_\mu$, (ii) $\mathcal{R}_{\mu/2}\mathcal{R}_{-\mu/2} = \mathcal{R}_{-\mu/2}\mathcal{R}_{\mu/2} = 1$ and (iii)$\Box = \sum_\mu (\mathcal{R}_{\mu/2} - \mathcal{R}_{-\mu/2})^2$. From the last property it follows their expression for the Dirac operator $\partial = \sum_\mu \gamma^\mu (\mathcal{R}_{\mu/2} - \mathcal{R}_{-\mu/2})$, where γ^μ are the generators of a Clifford algebra.

The problem in the FLS approach, as recognized by the authors, is the nature of the operator $\mathcal{R}_{\mu/2}$. According to their definition, these operators are of the same mathematical nature of the operators $\mathcal{R}_\mu$, and of course there is no sense in thinking of a half-space translation. The justification for their approach seems to be that it works. This is acceptable as an insight, but it must be justified mathematically. Our approach seems to provide such justification. Indeed, the essential point is the introduction of a square-root of

the translation operator, and in our approach it appeared naturally as those operators Γ^μ. While we were not able to see any mathematical justification for an operator like $\mathcal{R}_{\mu/2}$, this is not the case for the operators Γ^μ. The expressions in FLS approach are related to ours by the correspondence

$$\Gamma^\mu \leftrightarrow \gamma^\mu \mathcal{R}_{\mu/2}. \tag{58}$$

If we replace Γ^μ by $\gamma^\mu \mathcal{R}_{\mu/2}$ in the expression for the Dirac operator (57) we get exactly the expression used by FLS. However, in our oppinion this must not be taken as a justification for those operators $\mathcal{R}_{\mu/2}$ since this operator seems to us meaningless whatever approach we take. Anyway, the above correspondence strongly suggests that maybe we have here a possible solution to the old fermion doubling problem - in order to see how it works see [5].

6 Conclusions

We have shown how to construct a Dirac operator in complete analogy with the continuum over an abelian finite group. This was possible once we have introduced a new algebraic structure that generalizes the Clifford algebras (and which reduces to them in the continuum limit). The generators of this new Clifford-like algebra appear as square-root of the translation operators. These generators can be construct in exactly the same manner as in the continuum, from the operators of exterior (wedge) multiplication and (right) contraction. Moreover we still have the remarkable relation $\partial = \mathrm{d} - \delta$ involving the Dirac operator and the differential and codifferential operators. We have also shown how these results can be applied to lattice field theories, and in particular to a recent proposed solution to the fermion doubling problem, where that generalized Clifford-like algebra seems to provide its mathematical basis.

Acknowledgments. We are grateful to Professor A. P. Balachandran and A. Zajec for their kind hospitality at SU and to FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo - Brazil) for the finnancial support.

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